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PIECEWISE-POLYNOMIAL APPROXIMATION OF FUNCTIONS FROM $H^\ell((0,1)^d)$, $2\ell = d$, AND APPLICATIONS TO THE SPECTRAL THEORY OF THE SCHRÖDINGER OPERATOR

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ABSTRACT

For the selfadjoint Schrödinger operator $-\Delta - \alpha V$ on $\mathbb{R}^2$ the number of negative eigenvalues is estimated. The estimates obtained are based upon a new result on the weighted L_2-approximation of functions from the Sobolev spaces in the cases corresponding to the critical exponent in the embedding theorem.

0. Introduction

This paper is devoted to the analysis of two closely related problems. The first one is the problem of the weighted L_2- approximation of functions from the Sobolev space $H^\ell(Q)$ by means of piecewise-polynomial functions; here Q is the unit cube in $\mathbb{R}^d$ and the crucial point is that we are dealing with the case $2\ell = d$. The main aim here is to obtain uniform estimates of approximation with respect to the widest class of weights allowed by the embedding theorems. Such a problem was investigated earlier in [BS1], [R1], [R2] for the cases $2\ell > d$, $2\ell < d$ (see also a unified exposition in [BS2]). The sharp version of the embedding theorem for $2\ell = d$ involves some Orlicz spaces and it is not quite clear how to extend the approach proposed in the above papers to this case. This difficulty is precisely what is overcome here. We introduce a new (equivalent) norm in the Orlicz spaces $L_\Psi(E)$, $E \subset \mathbb{R}^d$, in such a way that it satisfies some nice invariance properties with respect to scaling. Using this norm one can adapt the approach of [R1],

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[R2] to the case of Orlicz spaces. The main result on approximations, Theorem 1, is proved in Section 1.

The second problem considered concerns estimates on the number of the negative eigenvalues of the Schrödinger operator $-\Delta - \alpha V$ on $\mathbb{R}^2$. We are interested in estimates sharp in α, $\alpha \to \infty$, and as far as possible, also in function classes for V. A general scheme of applying piecewise-polynomial approximation to such problems was developed long ago by Birman and the author; its detailed exposition was given in [BS2]. Sharp estimates of the desired type are known for $d \geq 3$ (the so-called Rozenblum–Cwikel–Lieb estimate, see [R1], [R2], [C], [L1], [L2], [L3], and also expositions in [BS2], [S2], [RS]) and for $d = 1$ ([BB], [BS3]). Theorem 1 allows us to get satisfactory results also for $d = 2$. The former results for $d = 2$ [BB] were rather incomplete because they were based upon a much more restrictive result from [BS1] than the one given by Theorem 1.

The material concerning these estimates is presented in Sections 2 and 3. In order to avoid making this paper too long, we restrict ourselves to the investigation of the Schrödinger operator only. Correspondingly, we deal only with the simplest case $2\ell = d = 2$ of Theorem 1.

The concluding Section 4 contains a detailed commentary and discussion of the results obtained.

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1. Main result on approximations

1. Here we consider the approximation of functions $u \in H^\ell(Q)$, where $Q = Q^d = (0,1)^d$, $2\ell = d$, by means of piecewise-polynomial functions. The construction of the approximating function is related to an appropriate choice of a covering Ξ of Q by parallelepipeds $\Delta \subset \mathbb{R}^d$. We always consider parallelepipeds $\Delta \subset \mathbb{R}^d$ (as a rule, cubes) with the edges parallel to the ones of Q.

We now introduce some notations. By $\mathcal{P}(\ell, d)$ we denote the linear space of all polynomials of degree $< \ell$ in $\mathbb{R}^d$, by $m(\ell, d)$ – its dimension. For any parallelepiped $\Delta \subset \mathbb{R}^d$ we can regard $\mathcal{P}(\ell, d)$ as a subspace in $L_2(\Delta)$; let $P_{\Delta,\ell}$ be the corresponding orthogonal projection.

Furthermore, let Ξ be a finite covering of Q by parallelepipeds $\Delta \subset Q$. To any such covering and any $\ell > 0$ we associate an operator of piecewise-polynomial approximation. Namely, we enumerate the parallelepipeds $\Delta \in \Xi$ in some way, $\Xi = \{\Delta_j\}$, $1 \leq j \leq \text{card}(\Xi)$. Denote by χ_j the characteristic function of the set $\Delta_j \setminus \cup_{i<j} \Delta_i$. The approximation operator mentioned is

$$K_{\Xi,\ell} = \sum_j \chi_j P_{\Delta_j,\ell} \; . \tag{1}$$

Evidently,

$$\text{rank } K_{\Xi,\ell} \leq m(\ell, d)\text{card}(\Xi) \; . \tag{2}$$

We need also some knowledge in Orlicz spaces (see [KR] and [A], Chapter VIII). Let Φ, Ψ be mutually complementary N-functions, $L_\Phi(E), L_\Psi(E)$ be the corresponding Orlicz spaces on a set $E \subset \mathbb{R}^d$ of finite Lebesgue measure $|E|$. Along with the classical Orlicz norm on $L_\Psi(E)$

$$\|v\|_{\Psi,E} = \sup\{|\int_E vf dx|: \int_E \Phi(f(x))dx \leq 1\} \; , \tag{3}$$

we introduce the "average Orlicz norm"

$$\|v\|^{(av)}_{\Psi,E} = \sup\{|\int_E vf dx|: \int_E \Phi(f(x))dx \leq |E|\} \; . \tag{4}$$

The norms (3) and (4) are mutually equivalent but the coefficients in the corresponding two-sided inequality depend on $|E|$. Clearly $\|v\|^{(av)}_{\Psi,E} = \|v\|_{\Psi,E}$ when $|E| = 1$. In fact we only need the pair

$$\mathcal{A}(t) = e^{|t|} - 1 - |t| \; , \qquad \mathcal{B}(t) = (1 + |t|)\ell n(1 + |t|) - |t| \; ,$$

but the auxiliary results obtained below for the general case may be of some independent interest.

We are now in a position to formulate our main result on approximations. In what follows

$$|\nabla_\ell u|^2 := \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} |\mathcal{D}^\alpha u|^2.$$

We denote by C_n the constants which are repeated; number n indicates the formula where this constant first appears.

THEOREM 1: *Let $Q = (0,1)^d$, $V \in L_{\mathcal{B}}(Q)$, $V \geq 0$. Then for any $n \in \mathbb{N}$ there exists a covering $\Xi = \Xi(V,n)$ of Q by parallelepipeds $\Delta \subset Q$ such that*

$$\operatorname{card}(\Xi) \leq C_5 n \tag{5}$$

and for any function $u \in H^\ell(Q)$, $2\ell = d$, we have

$$\int_Q V|u - K_{\Xi,\ell}u|^2 dx \leq C_6 n^{-1} \|V\|_{\mathcal{B},Q} \int_Q |\nabla_\ell u|^2 dx \tag{6}$$

where $K_{\Xi,\ell}$ is the operator introduced by (1). The constants C_5, C_6 depend only on d (but not on V).

In order to prove the Theorem, we need some more information on Orlicz spaces and on coverings. This information is collected in Subsection 2.

2. The norm (4) is, in a sense, invariant with respect to scaling:

LEMMA 1: *Let ξ be an affine transformation of $\mathbb{R}^d$ and $E_\xi = \xi(E)$. Then for any N-function Ψ and any $v \in L_\Psi(E_\xi)$*

$$|E|^{-1}\|v \circ \xi\|^{(av)}_{\Psi,E} = |E_\xi|^{-1}\|v\|^{(av)}_{\Psi,E_\xi} . \tag{7}$$

We omit the proof consisting of a standard change of variables in both integrals in (4).

The well-known embedding theorem concerning the Sobolev space with the critical exponent (see [A], Theorem 8.25) will be used in the sequel. For the case $E = Q$ it can be written in the following form.

PROPOSITION 1: *There exists a constant $C_8(d)$ such that for every $u \in H^\ell(Q)$, $Q = (0,1)^d$, $2d = \ell$*

$$\|u^2\|_{\mathcal{A},Q} \leq C_8 \|u\|^2_{H^\ell(Q)} . \tag{8}$$

On the subspace $Ker\ P_{Q,\ell}$ of $H^\ell(Q)$ the norm $\|u\|_{H^\ell}$ is equivalent to $\|\nabla_\ell u\|_{L_2}$. It follows from this and from (8) that, with some other constant factor,

$$\|u^2\|_{\mathcal{A},Q} \leq C_9(d) \int_Q |\nabla_\ell u|^2 dx , \quad u \in H^\ell(Q) , \quad P_{Q,\ell}u = 0 , \quad 2\ell = d . \tag{9}$$

Combining this estimate with Lemma 1, we obtain the following useful result.

LEMMA 2: *Let $\Delta \subset \mathbb{R}^d$ be a parallelepiped with edges of length $h_1, \ldots, h_d$ and $u \in H^\ell(\Delta)$, $2\ell = d$, $P_{\Delta,\ell}u = 0$. Then for any non-negative function $V \in L_\mathcal{B}(\Delta)$*

$$(10) \qquad \int_\Delta V|u|^2 dx \le C_9(d) \max_{i,j} \left(\frac{h_i}{h_j}\right)^d \|V\|^{(av)}_{\mathcal{B},\Delta} \int_\Delta |\nabla u|^2 dx \ .$$

Proof: If $\Delta = Q$, then (10) is a direct consequence of the Hölder inequality for Orlicz spaces (see [KR], Theorem 9.3) and of (9):

$$\int_Q V|u|^2 dx \le \|V\|_{\mathcal{B},Q} \|u^2\|_{\mathcal{A},Q} \le C_9(d)\|V\|_{\mathcal{B},Q} \int_Q |\nabla_\ell u|^2 dx.$$

For an arbitrary $\Delta \subset \mathbb{R}^d$, let ξ be an affine transformation of $\mathbb{R}^d$ such that $\xi(Q) = \Delta$. If $u \in H^\ell(\Delta)$, then $u \circ \xi \in H^\ell(Q)$ and, due to the condition $2\ell = d$,

$$(11) \qquad \int_Q |\nabla_\ell(u \circ \xi)|^2 dx \le \max_{i,j} \left(\frac{h_i}{h_j}\right)^d \int_\Delta |\nabla_\ell u|^2 dx \ .$$

Besides, $P_{Q,\ell}(u \circ \xi) = 0$ provided $P_{\Delta,\ell}u = 0$. By (9) and (11),

$$\int_\Delta V|u|^2 dx = |\Delta| \int_Q V \circ \xi |u \circ \xi|^2 dx$$

$$\le C_9(d) \max_{i,j} \left(\frac{h_i}{h_j}\right)^d |\Delta| \|V \circ \xi\|_{\mathcal{B},Q} \int_\Delta |\nabla_\ell u|^2 dx \ .$$

Taking into account (7), we see that this coincides with the required inequality (10). ∎

Note that for the case of cubes, the inequality (11) turns into equality. Moreover, for any homothety transformation ξ and for any region $E \subset \mathbb{R}^d$ we have

$$(12) \qquad \int_E |\nabla_\ell(u \circ \xi)|^2 dx = \int_{\xi(E)} |\nabla_\ell u|^2 dx \ , \quad \forall u \in H^\ell(\xi(E)) \ , \quad 2\ell = d \ .$$

Now we return again to the case of an arbitrary N-function Ψ. For a fixed $v \in L_\Psi(E)$, we will investigate some properties of the quantity

$$(13) \qquad \mathcal{J}(e) = \mathcal{J}_{\Psi,v}(e) = \|v\|^{(av)}_{\Psi,e}$$

as a function of measurable sets $e \subset E$.

LEMMA 3: *The function (13) is superadditive, i.e., for any $e \subset E$ and any finite system of mutually disjoint subsets $e_j \subset e$*

$$\sum_j \mathcal{J}(e_j) \le \mathcal{J}(e).$$

Proof: Let Φ be the N-function complementary to Ψ. With any function f_j on e_j satisfying the inequality

$$\int_{e_j} \Phi(f_j(x))dx \le |e_j| \tag{14}$$

we associate the function F_j on e: $F_j = f_j$ on e_j, $F_j = 0$ on $e \backslash e_j$. Then for

$$F = \sum_j F_j \tag{15}$$

we have

$$\int_e \Phi(F(x))dx = \sum_j \int_{e_j} \Phi(f_j(x))dx \le \sum_j |e_j| \le |e|. \tag{16}$$

At the same time

$$\int_e v(x)F(x)dx = \sum_j \int_{e_j} v(x)f_j(x)dx. \tag{17}$$

It follows from (14), (16), (17) and from the definition (4) that

$$\begin{aligned} \|v\|^{(av)}_{\Psi,e} &= \sup\{|\int_e v(x)F(x)dx|: \int_e \Phi(F(x))dx \le |e|\} \\ &\ge \sup\{|\int_e v(x)F(x)dx|: F \text{ of the form (15)}\} \\ &= \sum_j \sup\{|\int_{e_j} v(x)f_j(x)dx|: \int_{e_j} \Phi(f_j(x))dx \le |e_j|\} \\ &= \sum_j \|v\|^{(av)}_{\Psi,e_j}. \quad \blacksquare \end{aligned}$$

Let us suppose now that Ψ satisfies the Δ_2-condition (see [KR], §4). For this case and for $E = Q$ we can prove that the function $\mathcal{J}_{\Psi,v}$ is, in a sense, continuous. Recall that the class $C(\bar{Q})$ is dense in $L_\Psi(Q)$ for any such Ψ (see [A], Theorem 8.20).

For a given point $x \in \bar{Q}$ and $t > 0$, let $\Delta_x(t)$ be the cube centered in x, with the edges of length t. Denote $\tilde{\Delta}_x(t) = \Delta_x(t) \cap Q$ and consider the function $j(t) = \mathcal{J}_{\Psi,v}(\tilde{\Delta}_x(t))$, $t > 0$. It is clear that for every $x \in \bar{Q}$, $j(t)$ is a non-decreasing function and $j(t) = \mathcal{J}_{\Psi,v}(Q)$ for $t \ge 2$.

LEMMA 4: *Suppose that the Δ_2-condition is satisfied for Ψ. Then $j(t)$ is continuous and $j(0+) = 0$.*

Proof: For a given t_0, $t > 0$, let $\xi_{t_0,t}$ be an affine transformation of $\mathbb{R}^d$ such that $\xi_{t_0,t}(\tilde{\Delta}_x(t_0)) = \tilde{\Delta}_x(t)$. Then by (7)

$$j(t) = \frac{|\tilde{\Delta}_x(t)|}{|\tilde{\Delta}_x(t_0)|}\|v \circ \xi_{t_0,t}\|^{(av)}_{\Psi,\tilde{\Delta}_x(t_0)}. \tag{18}$$

If $v \in C(\bar{Q})$, then $v \circ \xi_{t_0,t} \to v$ uniformly on $\tilde{\Delta}_x(t_0)$ as $t \to t_0$. Therefore

$$\int_{\tilde{\Delta}_x(t_0)} \Psi(v(\xi_{t_0,t}(x)) - v(x))dx \longrightarrow 0$$

and, because of the Δ_2-condition for Ψ,

$$\|v \circ \xi_{t_0,t} - v\|^{(av)}_{\Psi,\tilde{\Delta}_x(t_0)} \longrightarrow 0 \tag{19}$$

([KR], Theorem 9.4). By continuity, (19) remains valid for any $v \in L_\Psi(Q)$. It follows that $\|v \circ \xi_{t_0,t}\|^{(av)}_{\Psi,\tilde{\Delta}_x(t_0)} \longrightarrow \|v\|^{(av)}_{\Psi,\tilde{\Delta}_x(t_0)}$ as $t \to t_0$ and it is clear from (18) that $j(t) \to j(t_0)$.

The property $j(+0) = 0$ is a straightforward consequence of the absolute continuity of the norm in L_Ψ, see [KR], Theorem 10.3. ∎

We will use the Besicovitch covering lemma when proving Theorem 1. To formulate it, we need a notion of the "linkage" of a covering Ξ of $\bar{Q}$ by cubes $\Delta \subset \mathbb{R}^d$. Suppose that Ξ can be split into r subsets $\Xi_1, \ldots, \Xi_r$ in such a way that for each $k = 1, \ldots, r$ the cubes $\Delta \in \Xi_k$ are pairwise disjoint. The smallest number r for which such a partition of Ξ is possible is called the *linkage* of Ξ and denoted by $link(\Xi)$.

PROPOSITION 2 (Besicovitch covering lemma; see [G], Theorem 1.1): *Let for any point $x \in \bar{Q} = [0,1]^d$, a closed cube $\Delta_x \subset \mathbb{R}^d$ centered in x, be given. Then a subset $\Xi = \{\Delta_{x_j}\}$ can be chosen in such a way that $\cup_j \Delta_{x_j} \supset \bar{Q}$ and $link(\Xi) \leq r_d$ where r_d is a number depending only on the dimension d.*

3. PROOF OF THEOREM 1..

In the course of the proof $\mathcal{J}(e)$ is the function (13) corresponding to $\Psi = \mathcal{B}$ and $v = V$. Note that the Δ_2-condition is satisfied for $\mathcal{B}$. We can normalize V assuming $\mathcal{J}(Q) = \|V\|_{\mathcal{B},Q} = 1$.

Fix $n \in \mathbb{N}$ and for each $x \in \bar{Q}$ find a cube Δ_x centered in x such that $\|V\|^{(av)}_{\mathcal{B},\Delta_x \cap Q} = n^{-1}$. The existence of such a cube follows from Lemma 4. Let

Ξ be the covering of $\bar{Q}$ selected according to Proposition 2, and $\Xi = \cup_k \Xi_k$, $1 \le k \le link(\Xi)$, be any partition of Ξ appearing in the definition of linkage. Then, by the superadditivity of $\mathcal{J}$ (Lemma 3)

$$n^{-1}\mathrm{card}(\Xi_k) = \sum_{\Delta \in \Xi_k} \mathcal{J}(\Delta \cap Q) \le \mathcal{J}(Q) = 1,$$

hence

$$\mathrm{card}(\Xi) \le n\ link(\Xi) \le n r_d. \tag{20}$$

The parallelepipeds $\tilde{\Delta} = \Delta \cap Q$, $\Delta \in \Xi$, constitute a covering $\tilde{\Xi}$ of Q. If h_i, $i = 1, \ldots, d$, are the lengths of the edges of $\tilde{\Delta} \in \tilde{\Xi}$, then $\max_{i,j} \frac{h_i}{h_j} \le 2$ (because Δ is centered in $x \in \bar{Q}$).

Let $K_{\tilde{\Xi},\ell}$ be the approximation operator given by (1). Then, by (2) and (20),

$$\mathrm{rank}\ K_{\tilde{\Xi},\ell} \le n m(\ell, d) r_d. \tag{21}$$

For any $u \in H^\ell(Q)$ we evidently have

$$\int_Q V|u - K_{\tilde{\Xi},\ell} u|^2 dx \le \sum_{\tilde{\Delta} \in \tilde{\Xi}} \int_{\tilde{\Delta}} V|u - P_{\tilde{\Delta},\ell} u|^2 dx. \tag{22}$$

We can apply the estimate (10) to every summand of the last sum because $P_{\tilde{\Delta},\ell}(u - P_{\tilde{\Delta},\ell} u) = 0$ and get

$$\int_Q V|u - K_{\tilde{\Xi},\ell} u|^2 dx \le 2^d C_9 \sum_{\tilde{\Delta} \in \tilde{\Xi}} \mathcal{J}(\tilde{\Delta}) \int_{\tilde{\Delta}} |\nabla_\ell u|^2 dx$$

$$\le 2^d C_9 n^{-1} \sum_{\tilde{\Delta} \in \tilde{\Xi}} \int_{\tilde{\Delta}} |\nabla_\ell u|^2 dx$$

$$= 2^d C_9 n^{-1} \sum_k \left(\sum_{\Delta \in \Xi_k} \int_{\Delta \cap Q} |\nabla_\ell u|^2 dx \right) \le 2^d C_9 r_d n^{-1} \int_Q |\nabla_\ell u|^2 dx\ .$$

This coincides with the conclusion of Theorem 1, with $C_5 = r_d$ and $C_6 = 2^d C_9 r_d$. ∎

2. Applications to the spectral theory

1. As was mentioned in the Introduction, we restrict ourselves here to the case of $\ell = 1$, $d = 2$. Note that $m(1,2) = 1$. The main object we deal with is the Schrödinger operator in $L_2(\mathbb{R}^2)$

$$A_{\alpha V} = -\Delta - \alpha V, \tag{23}$$

with a non-negative potential V and coupling constant $\alpha > 0$. By definition, $A_{\alpha V}$ is the self-adjoint operator in $L_2(\mathbb{R}^2)$, associated with the quadratic form

$$a_{\alpha V}[u] = \int (|\nabla u|^2 - \alpha V |u|^2) dx \tag{24}$$

(here and in the sequel $\int := \int_{\mathbb{R}^2}$). It will follow from our assumptions on V that $a_{\alpha V}$ is bounded from below and closed on the domain $H^1(\mathbb{R}^2)$ and that the negative spectrum of the corresponding operator $A_{\alpha V}$ is discrete.

Recall the definitions of the spectrum distribution functions for unbounded and for compact operators in a Hilbert space. Suppose that the spectrum of a selfadjoint operator A to the left of a given point $\lambda \in \mathbb{R}$ is discrete. Then, by definition,

$$N(\lambda; A) = \operatorname{card}\{j \colon \lambda_j(A) < \lambda\},$$

where $\lambda_j(A)$ are the eigenvalues of A counted according to their multiplicities. Quite similarly, for a compact, non-negative symmetric operator T,

$$n(\lambda; T) := N(-\lambda, -T) = \operatorname{card}\{j \colon \lambda_j(T) > \lambda\}\ , \quad \lambda > 0.$$

Our main goal is to get some estimates of the quantity $N(-\gamma; A_{\alpha V})$ for the operator (23) and $\gamma \geq 0$. The estimates for $\gamma > 0$ and for $\gamma = 0$ look quite different, the latter one being much more involved. We now formulate the corresponding statements.

For any $h > 0$, let $\{Q_k(h)\}$, $k \in \mathbb{Z}^2$, be the lattice of squares in $\mathbb{R}^2$ with edges of length h. We write Q_k instead of $Q_k(1)$. Q without an index denotes the standard unit square.

THEOREM 2: *Let $V \in L_{\mathcal{B},loc}(\mathbb{R}^2)$ and suppose that the series $\sum_{k\in\mathbb{Z}^2} \|V\|_{\mathcal{B},Q_k}$ converges. Then for any $\alpha > 0$ the quadratic form (24) is bounded from below* *and closed on $H^1(\mathbb{R}^2)$, the negative spectrum of the corresponding operator (23) is discrete and for any $\gamma > 0$ the following estimate is valid:*

$$N(-\gamma, A_{\alpha V}) \le C_{25}\alpha \sum_{k\in\mathbb{Z}^2} \|V\|^{(av)}_{\mathcal{B},Q_k(\gamma^{-\frac{1}{2}})}. \tag{25}$$

To formulate the result for $\gamma = 0$, we need some additional notations. Introduce two different partitions of $\mathbb{R}^2$:

$$\mathbb{R}^2 = \bigcup_{i\ge 0} \Omega_i = \bigcup_{i\ge 0} \Theta_i$$

where

$$\begin{aligned} &\Omega_0 = \{x: |x| \le 1\}, \quad \Omega_i = \{x: 2^{i-1} \le |x| \le 2^i\}, \quad i \in \mathbb{N}\ ; \\ &\Theta_0 = \{x: |x| \le e\}, \quad \Theta_i = \{x: 2^{i-1} \le \ell n|x| \le 2^i\}, \quad i \in \mathbb{N}\ . \end{aligned} \tag{26}$$

To a given potential V, we correspond two numerical sequences, $\mu(V) = \{\mu_i(V)\}$, $\nu(V) = \{\nu_i(V)\}$, $i \ge 0$:

$$\mu_i(V) = \|V\|^{(av)}_{\mathcal{B},\Omega_i}\ , \tag{27}$$

$$\nu_i(V) = \int_{\Theta_i} V(x)|\ell n|x||dx\ . \tag{28}$$

Note that

$$\nu_0(V) \le \mu_0(V) + \mu_1(V) + \mu_2(V). \tag{29}$$

Indeed, it is easy to check that $\int_{\Omega_i} \mathcal{A}(\ell n|x|)dx \le |\Omega_i|$ for $i = 0, 1, 2$, and (29) follows from the definition (4), due to the inclusion $\Theta_i \subset \Omega_0 \cup \Omega_1 \cup \Omega_2$.

In fact each term $\nu_i(V)$ can be estimated by $\sum_j \mu_j(V)$, where summation extends over all j such that $\Theta_i \cap \Omega_j \ne \emptyset$, but with a constant factor depending on i.

Recall that a sequence $\eta = \{\eta_i\}$ belongs to the "weak ℓ_1-space" $\ell_{1,w}$ if the following quantity (the quasinorm of η) is finite:

$$\|\eta\|_{1,w} = \sup_{s>0}(s\ \mathrm{card}\{i: |\eta_i| > s\}).$$

It is well known that $\ell_1 \subset \ell_{1,w}$ and

$$\|\eta\|_{1,w} \le \|\eta\|_1. \tag{30}$$

THEOREM 3: *Let $V \in L_{B,loc}(\mathbb{R}^2)$ and $\mu(V) \in \ell_1$, $\nu(V) \in \ell_{1,w}$. Then for any $\alpha > 0$ the quadratic form (24) is bounded from below and closed on $H^1(\mathbb{R}^2)$, the negative spectrum of the corresponding operator (23) is finite and the following estimate is valid:*

$$N(0; A_{\alpha V}) \leq 1 + C_{31}\alpha(\|\mu(V)\|_1 + \|\nu(V)\|_{1,w}) . \tag{31}$$

Note that neither of the terms appearing in (31) can be estimated by the other one. The estimate (31) looks rather inconvenient because of the presence of two dissimilar terms on its right hand side. The following estimate which follows from (31) and (30) may turn out to be more efficient:

$$N(0; A_{\alpha V}) \leq 1 + C_{31}\alpha(\|\mu(V)\|_1 + \int V(x)|\ell n|x||dx). \tag{32}$$

2. The rest of the paper is devoted to the proof of Theorems 2 and 3. First, we derive some spectrum estimates for operators associated with a class of variational problems.

Let $\Omega \subseteq \mathbb{R}^2$ be a region. On the space $H^1(\Omega)$ endowed with the standard Sobolev norm, we consider the quadratic functional

$$b_{\Omega,V}[u] = \int_\Omega V|u|^2 dx, \tag{33}$$

where V is a given measurable non-negative function on Ω. If $b_{\Omega,V}$ is bounded on $H^1(\Omega)$, then it generates a bounded self-adjoint, non-negative operator – say $T_{\Omega,V}$ – in this space. Recall that, by definition, for a given $f \in H^1(\Omega)$,

$$u = T_{\Omega,V} f \Leftrightarrow u \in H^1(\Omega) ;$$

$$\int_\Omega (\nabla u \cdot \nabla \bar{v} + u\bar{v})dx = \int_\Omega V f \bar{v} dx , \quad \forall v \in H^1(\Omega) .$$

Under some assumptions on Ω and V, the operator $T_{\Omega,V}$ turns out to be compact and we are interested in the behavior of the distribution function $n(\lambda; T_{\Omega,V})$ as $\lambda \to +0$.

THEOREM 4: *Let $\Omega \subset \mathbb{R}^2$ be a bounded region with a Lipschitzian boundary and $V \in L_B(\Omega)$. Then the operator $T_{\Omega,V}$ is compact and there exists a constant $C_{34}(\Omega)$ such that for any $\lambda > 0$*

$$n(\lambda; T_{\Omega,V}) \leq C_{34}(\Omega)\|V\|_{B,\Omega}\lambda^{-1} . \tag{34}$$

Proof: What we give below is quite standard. In fact, it almost coincides with the proof of Theorem 4.1 from [BS2] for the case $2\ell = d = 2$. The only but decisive distinction is that we are now in a position to apply Theorem 1 instead of a less precise statement used in [BS2].

Let $Q \subset \mathbb{R}^2$ be a square such that $\bar{\Omega} \subset Q$. We can regard Q as a unit square. Let $\tilde{V}$ be the function on Q equal V on Ω and $\tilde{V} = 0$ otherwise. Fix a bounded linear extension operator Π: $H^1(\Omega) \to H^1(Q)$. Using the Hölder inequality for the spaces $L_{\mathcal{A}}(Q), L_{\mathcal{B}}(Q)$ and the estimate (8), we get for any $u \in H^1(\Omega)$ and $U = \Pi u$:

$$\int_\Omega V|u|^2dx = \int_Q \tilde{V}|U|^2dx \le \|\tilde{V}\|_{\mathcal{B},Q}\|U^2\|_{\mathcal{A},Q}$$

$$\le C_8\|V\|_{\mathcal{B},\Omega}\|U\|^2_{H^1(Q)} \le C_8\|\Pi\|^2\|V\|_{\mathcal{B},\Omega}\|u\|^2_{H^1(\Omega)} \,.$$

Thus $T_{\Omega,V}$ is bounded and

$$n(\lambda; T_{\Omega,V}) = 0 \quad \text{for} \quad \lambda > C_8\|\Pi\|^2\|V\|_{\mathcal{B},\Omega} \,. \tag{35}$$

Now fix $\lambda \in (0, \lambda_0]$, where $\lambda_0 = C_6\|\Pi\|^2\|V\|_{\mathcal{B},\Omega}$. Let n be the minimal integer such that $n\lambda \ge \lambda_0$. For this n and the function $\tilde{V}$, let Ξ be the covering of Q constructed in Theorem 1 and $K_\Xi = K_{\Xi,1}$ be the corresponding operator (1). For the subspace $\mathcal{F} = Ker(K_\Xi\Pi)$ of $H^1(\Omega)$

$$\text{codim } \mathcal{F} \le \text{rank } K_\Xi \le r_2 n.$$

For $u \in \mathcal{F}$, the following inequality holds:

$$\int_\Omega V|u|^2dx = \int_Q \tilde{V}|U - K_\Xi U|^2dx \le C_6 n^{-1}\|\tilde{V}\|_{\mathcal{B},Q}\int_Q |\nabla U|^2dx$$

$$\le C_6 n^{-1}\|\Pi\|^2\|V\|_{\mathcal{B},\Omega}\|u\|^2_{H^1(\Omega)} \le \lambda\|u\|^2_{H^1(\Omega)} \,.$$

It follows that $T_{\Omega,V}$ is compact and, by the variational principle,

$$n(\lambda; T_{\Omega,V}) \le \text{codim } \mathcal{F} \le r_2 n < r_2(\frac{\lambda_0}{\lambda} + 1) \le 2r_2\frac{\lambda_0}{\lambda}\,, \quad \lambda \le \lambda_0. \tag{36}$$

The required estimate (34), with $C_{34} = 2r_2\|\Pi\|^2\max(C_6, C_8)$, is a direct consequence of (35) and (36). ∎

Consider now the subspace

$$\hat{H}^1(\Omega) = \{u \in H^1(\Omega): \int_\Omega u dx = 0\}.$$

We equip $\hat{H}^1(\Omega)$ with the norm $\|\nabla u\|_{L_2}$ which is equivalent on this subspace to the standard norm $\|u\|_{H^1}$. Let $\hat{T}_{\Omega,V}$ be the selfadjoint operator in $\hat{H}^1(\Omega)$ generated by the same quadratic functional (33). For $\hat{T}_{\Omega,V}$ the estimate (34) clearly remains true, in general with some other constant factor. A significant difference between the estimates for $T_{\Omega,V}$ and $\hat{T}_{\Omega,V}$ is expressed by the following statement.

THEOREM 4′: *Under the assumptions of Theorem 4,*

$$n(\lambda; \hat{T}_{\Omega,V}) \leq C_{37}(\Omega)\|V\|_{\mathcal{B},\Omega}^{(av)}\lambda^{-1}, \quad \forall\lambda > 0. \tag{37}$$

If Ω', Ω'' are homothetic regions, then

$$C_{37}(\Omega') = C_{37}(\Omega''). \tag{38}$$

Proof: We only need to check (38). For this aim, it suffices to use the scaling $x \mapsto hx$ and to take into account the invariance properties (12) of $\|\nabla u\|_{L_2}$ and (7) of the norm (4). ∎

3. Here the case of $\Omega = \mathbb{R}^2$ is considered. Theorems 4 and 4′ have no direct analogues in this case, but it is easy to derive some eigenvalue estimates using suitable partitions of $\mathbb{R}^2$ into a collection of bounded regions.

The statement given below is a consequence of Theorem 4. A similar consequence of Theorem 4′ will be stated in §3.

COROLLARY 1: *Let $V \in L_{\mathcal{B},loc}(\mathbb{R}^2)$ and suppose that the series $\sum_{k\in\mathbb{Z}^2}\|V\|_{\mathcal{B},Q_k}$ converges. Then the operator $T_{\mathbb{R}^2,V}$ is compact and*

$$n(\lambda; T_{\mathbb{R}^2,V}) \leq C_{34}(Q)\lambda^{-1}\sum_{k\in\mathbb{Z}^2}\|V\|_{\mathcal{B},Q_k}, \qquad \lambda > 0.$$

Proof: By the variational principle,

$$n(\lambda; T_{\mathbb{R}^2,V}) \leq \sum_{k\in\mathbb{Z}^2} n(\lambda; T_{Q_k,V}).$$

It remains to apply Theorem 4 to each term of the last sum. ∎

4. Theorem 2 can be easily reduced to Corollary 1 with the help of the well known "Birman–Schwinger principle". Its general operator-theoretic formulation was proposed by Birman [B]. In connection with Proposition 3 given below, see [B], Theorem 1.4, or an exposition in [BS3], §1.

Let $a[u]$ be a positive ($a[u] > 0$ for $u \neq 0$) and closed quadratic form in a Hilbert space $\mathcal{H}$, with the dense domain $\mathrm{d} = \mathrm{d}[a]$. Let $b[u]$ be another non-negative quadratic form subject to the condition

$$b[u] \leq Ca[u] \ , \qquad u \in \mathrm{d}. \tag{39}$$

Consider a new Hilbert space $\tilde{\mathrm{d}}$ – the completion of d in the a-metric $a[u]$. It follows from (39) that b can be extended by continuity to the whole of $\tilde{\mathrm{d}}$. The extended form defines on $\tilde{\mathrm{d}}$ a bounded selfadjoint and non-negative operator – say B.

PROPOSITION 3: *Suppose that (39) is satisfied and the operator B is compact in $\tilde{d}$. Then for any $\alpha > 0$ the quadratic form*

$$a_\alpha[u] = a[u] - \alpha b[u] \ , \qquad u \in d \ ,$$

is bounded from below and closed in $\mathcal{H}$. For the self-adjoint operator A_α associated with this form, the negative spectrum is finite and

$$N(0; A_\alpha) = n(\alpha^{-1}, B). \tag{40}$$

Proof of Theorem 2: For the leading case $\gamma = 1$, (25), with $C_{25} = C_{34}(Q)$, is a straightforward consequence of Corollary 1 and of the identity

$$N(-1, A_{\alpha V}) = n(\alpha^{-1}, T_{\mathbb{R}^2, V}) \ , \quad \forall \alpha > 0.$$

In turn, this identity is nothing but a specific case of (40): we have to take

$$a[u] = \int (|\nabla u|^2 + |u|^2) dx, \ \tilde{\mathrm{d}} = \mathrm{d} = H^1(\mathbb{R}^2), \ b[u] = \int V|u|^2 dx.$$

Then we pass on from $\gamma = 1$ to any $\gamma > 0$ with the help of scaling $x \mapsto \gamma^{\frac{1}{2}} x$. ∎

In fact we were dealing with a very simple case when proving Theorem 2. It would be easy here to give a direct proof avoiding formal references to Proposition 3. The case of $\gamma = 0$ considered in the next section is much more delicate.

3. Proof of Theorem 3

1. To prove Theorem 3, we need some new objects. First, consider some Hilbert function spaces on $\mathbb{R}^2$. Sometimes we use the "naive" notation $f(x) = f(r, \varphi)$, where r, φ are the polar coordinates on $\mathbb{R}^2$. Let

$$\hat{f}_m(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \varphi) e^{-im\varphi} d\varphi , \qquad m \in \mathbb{N},$$

be the Fourier coefficients of f with respect to the φ-variable. With the help of the Fourier series

$$f(x) = f(r, \varphi) = \sum_{m \in \mathbb{Z}} \hat{f}_m(r) e^{im\varphi}$$

we introduce the following subspaces of $L_2(\mathbb{R}^2)$ and of $H^1(\mathbb{R}^2)$:

$$F_0 = \{f \in L_2\colon f(x) = \hat{f}_0(r)\}, \;\; F_1 = \{f \in L_2\colon \hat{f}_0(r) = 0\}, \tag{41}$$

$$G_0 = \{f \in H^1\colon f(x) = \hat{f}_0(r)\}, \;\; G_1 = \{f \in H^1\colon \hat{f}_0(r) = 0\}. \tag{42}$$

It is clear that these subspaces are closed, G_i is dense in F_i ($i = 0, 1$) and that the decompositions

$$L_2(\mathbb{R}^2) = F_0 \oplus F_1 , \qquad H^1(\mathbb{R}^2) = G_0 \oplus G_1$$

are orthogonal with respect to the corresponding scalar products. The following Hardy-type inequality is valid on G_1:

$$\begin{aligned} \int |\nabla f|^2 dx &= 2\pi \sum_{m \neq 0} \int_{\mathbb{R}_+} (|\hat{f}_m'|^2 + m^2 \frac{|\hat{f}_m|^2}{r^2}) r dr \\ &\geq 2\pi \sum_{m \neq 0} \int_{\mathbb{R}_+} |\hat{f}_m|^2 \frac{dr}{r} = \int |f|^2 \frac{dx}{|x|^2} , \quad f \in G_1 . \end{aligned} \tag{43}$$

Denote by $\mathcal{G}_1$ the completion of G_1 in the metric generated by $\int |\nabla f|^2 dx$. It follows from (43) that $\mathcal{G}_1$ is a function space embedded into the weighted space $L_2(\mathbb{R}^2; |x|^{-2})$. Note that the completion of G_0 in the metric $\int |\nabla f|^2 dx$ can not be realized as a function space on $\mathbb{R}^2$.

Denote by $a_{\alpha V}^{(i)}$, $i = 0, 1$, the restriction on G_i of the quadratic form given by (24), and by $A_{\alpha V}^{(i)}$, the self-adjoint operator in F_i associated with $a_{\alpha V}^{(i)}$. Writing $f = f_0 + f_1$, where $f \in H^1(\mathbb{R}^2)$, $f_0 \in G_0$, $f_1 \in G_1$, we get

$$\int V |f|^2 dx \leq 2 \int V |f_0|^2 dx + 2 \int V |f_1|^2 dx. \tag{44}$$

It follows from (44) by the standard variational arguments that

$$N(0; A_{\alpha V}) \le N(0; A^{(0)}_{2\alpha V}) + N(0; A^{(1)}_{2\alpha V}). \tag{45}$$

Thus it is sufficient to examine the operators $A^{(0)}_{\alpha V}, A^{(1)}_{\alpha V}$.

2. It is convenient for us to consider $A^{(1)}_{\alpha V}$ first.

Keeping in mind that we intend to apply Proposition 3, we introduce the self-adjoint operator S_V in $\mathcal{G}_1$, generated by the quadratic functional $\int V|f|^2dx$. In the following statement the notations Ω_i, $\mu(V) = \{\mu_i(V)\}$ from (26) and (27) are used.

COROLLARY 1′ (of Theorem 4′): *Let $V \in L_{\mathcal{B},loc}(\mathbb{R}^2)$ and $\mu(V) \in \ell_1$. Then the operator S_V is compact and*

$$n(\lambda; S_V) \le C_{46}\lambda^{-1}\|\mu(V)\|_1 , \qquad C_{46} = \max(C_{37}(\Omega_0), C_{37}(\Omega_1)). \tag{46}$$

Proof: The proof is quite similar to the one of Corollary 1. By the variational principle,

$$n(\lambda; S_V) \le \sum_{i\ge 0} n(\lambda; \hat{T}_{\Omega_i,V}).$$

(Note that here we not only eliminate the compatibility conditions on $\partial\Omega_i$ for the functions from $\mathcal{G}_1$ but also replace $\mathcal{G}_1|_{\Omega_i}$ by the much wider spaces $\hat{H}^1(\Omega_i)$.) Now (46) follows from Theorem 4′. ∎

Applying Proposition 3 (with $a[u] = \int|\nabla u|^2dx$, $\mathrm{d} = G_1$, $\tilde{\mathrm{d}} = \mathcal{G}_1$, $b[u] = \int V|u|^2dx$) we conclude that the form $a^{(1)}_{\alpha V}$ is bounded from below and closed in F_1. By (40) and (46),

$$N(0; A^{(1)}_{\alpha V}) \le C_{45}\alpha\|\mu(V)\|_1. \tag{47}$$

3. It remains to estimate $N(0; A^{(0)}_{\alpha V})$. This problem can be reduced to the one of estimating the number of negative eigenvalues for a Schrödinger operator on the axis $\mathbb{R}$.

After the standard substitution $r = e^t$, $u(x) = \hat{u}_0(r) = g(t)$ we get

$$\int|\nabla u|^2dx = 2\pi\int_{\mathbb{R}}|g'(t)|^2dt , \qquad \int|u|^2dx = 2\pi\int_{\mathbb{R}}|g(t)|^2e^{2t}dt ,$$

$$\int V|u|^2dx = 2\pi \int_{\mathbb{R}} W_V(t)|g(t)|^2dt\ , \tag{48}$$
$$\text{where } W_V(t) = \frac{e^{2t}}{2\pi}\int_0^{2\pi} V(e^t,\varphi)d\varphi\ .$$

This substitution maps G_0 onto the space

$$X = \{g\colon \int_{\mathbb{R}}(|g'|^2 + e^{2t}|g|^2)dt < \infty\}.$$

For $u \in G_0$ we have $a_{\alpha V}[u] = 2\pi \mathrm{a}_{\alpha W_V}[g]$, where

$$\mathrm{a}_{\alpha W_V}[g] = \int_{\mathbb{R}}(|g'|^2 - \alpha W_V|g|^2)dt. \tag{49}$$

Denote by $\mathrm{A}_{\alpha W_V}$ the self-adjoint operator in the weighted space $L_2(\mathbb{R}; e^{2t})$, associated with the quadratic form (49). It follows from the definition that $\mathrm{A}_{\alpha W_V}$ is unitarily equivalent to $(2\pi)^{-1}A^{(0)}_{\alpha V}$.

If we replace $L_2(\mathbb{R}; e^{2t})$ by $L_2(\mathbb{R})$ in the above definition, then we get a Schrödinger operator $-g'' - \alpha W_V g$ in $L_2(\mathbb{R})$. A scale of estimates of $N(0,\cdot)$ for such operators was obtained in [BS3], §6. Now we will show that these estimates remain valid for the operator $\mathrm{A}_{\alpha W_V}$ in $L_2(\mathbb{R}; e^{2t})$ as well.

Consider the space

$$\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}) = \{g \in H^1_{loc}(\mathbb{R})\colon g(0) = 0, \int_{\mathbb{R}} |g'|^2dt < \infty\}.$$

This is a Hilbert space with respect to the metric form $\int_{\mathbb{R}} |g'|^2dt$; due to the classical Hardy inequality, the integral $\int_{\mathbb{R}}(|g'|^2 + t^{-2}|g|^2)dt$ defines an equivalent metric in $\mathcal{H}^1$. Consider the quadratic functional $\int_{\mathbb{R}} W|g|^2dt$ and denote by M_W the self-adjoint operator in $\mathcal{H}^1$ generated by this functional. In [BS3], §6, the problem of estimating $n(\lambda; M_W)$ was analyzed (as a specific case $d = \ell = 1$ of a more general problem). A scale of estimates for $n(\lambda; M_W)$ was presented in an equivalent form, in Theorem 6.1.

In particular, for any given $q > \frac{1}{2}$ a sharp class of weight functions W was described guaranteeing the behavior $n(\lambda; M_W) = O(\lambda^{-q})$. For the corresponding Schrödinger operator $A_{\alpha W}$ this implies (due to (40)) the order $N(0; A_{\alpha W}) = O(\alpha^q)$. Clearly we need here such an estimate with $q = 1$ as the first term on the right hand side of (45) has just this order of growth.

We present an accurate formulation of the result we need. With a given potential $W \geq 0$ we associate the numerical sequence

$$\eta(W) = \{\eta_i(W)\}\ ,\quad i \in \mathbb{Z}\colon\ \eta_i(W) = \int_{2^{i-1}<|t|<2^i} |t|W(t)dt.$$

PROPOSITION 4: *Suppose that $\eta(W) \in \ell_{1,w}$. Then the operator M_W is compact and*

$$n(\lambda; M_W) \leq C_{50}\lambda^{-1}\|\eta(W)\|_{1,w}. \tag{50}$$

It is a little bit more convenient for us to deal with the sequences

$$\eta^{\pm}(W) = \{\eta_i^{\pm}(W)\}\ ,\quad i \in \mathbb{Z}\colon\ \eta_i^{\pm}(W) = \int_{2^{i-1}<\pm t<2^i} |t|W(t)dt$$

than with the initial sequence $\eta(W)$. It follows from (50) that

$$n(\lambda; M_W) \leq C_{51}\lambda^{-1}(\|\eta^+(W)\|_{1,w} + \|\eta^-(W)\|_{1,w}). \tag{51}$$

(The constant factor changes because $\|\cdot\|_{1,w}$ is a quasinorm but not a norm.)

For the potential W_V introduced by (48), we have

$$\eta_i^+(W_V) = \frac{1}{2\pi}\int_{e^{2^{i-1}}<|x|<e^{2^i}} V(x)\ell n|x|dx\ ,$$

$$\eta_i^-(W_V) = \frac{1}{2\pi}\int_{e^{-2^i}<|x|<e^{-2^{i-1}}} V(x)|\ell n|x||dx\ .$$

Thus,

$$\eta_i^+(W_V) = \frac{1}{2\pi}\nu_i(V)\ ,\qquad i > 0 \tag{52}$$

(recall that the numbers $\nu_i(V)$ were defined by (28)). Besides, by (30)

$$\begin{aligned} \|\eta^-(W_V)\|_{1,w} + \|\{\eta_i^+(W_V)\}_{i\leq 0}\|_{1,w} &\leq \sum_{i\in\mathbb{Z}}\eta_i^-(W_V) + \sum_{i\leq 0}\eta_i^+(W_V) \\ &= \frac{1}{2\pi}\int_{|x|<e} V(x)|\ell n|x||dx = \frac{1}{2\pi}\nu_0(V)\ . \end{aligned} \tag{53}$$

These calculations show that the conditions of Proposition 4 are satisfied for the potential W_V provided the assumptions of Theorem 3 are fulfilled for V, and therefore

$$n(\lambda; M_{W_V}) \leq \frac{C_{51}}{2\pi}(\nu_0(V) + \|\nu(V)\|_{1,w})\ . \tag{54}$$

Consider now the quadratic form (49) on the domain $X_0 = \{g \in X: g(0) = 0\}$. Note that

$$\dim(X/X_0) = 1. \tag{55}$$

It is easy to see that the completion of X_0 in the metric given by $\int_{\mathbb{R}} |g'|^2 dt$ coincides with $\mathcal{H}^1$. Applying Proposition 3 (with $a[g] = \int |g'|^2 dt$, $\mathrm{d} = X_0$, $\tilde{\mathrm{d}} = \mathcal{H}^1$, $b[g] = \int W_V |g|^2 dt$), we conclude that on X_0 the quadratic form (49) is bounded from below and closed in the space $L_2(\mathbb{R}, e^{2t})$. For the corresponding self-adjoint operator, say, $\tilde{\mathrm{A}}_{\alpha W_V}$, the following estimate flows out from (40) and (54):

$$N(0; \tilde{\mathrm{A}}_{\alpha W_V}) \leq \frac{C_{51}}{2\pi} \alpha(\nu_0(V) + \|\nu(V)\|_{1,w}).$$

This, together with (55), implies an estimate for the operator $A_{\alpha V}^{(0)}$:

$$\begin{aligned} N(0; A_{\alpha V}^{(0)}) &= N(0; \mathrm{A}_{\alpha W_V}) \leq 1 + N(0; \tilde{\mathrm{A}}_{\alpha W_V}) \\ &\leq 1 + \frac{C_{51}}{2\pi} \alpha(\nu_0(V) + \|\nu(V)\|_{1,w}) \ . \end{aligned} \tag{56}$$

Now the required estimate (31), with $C_{31} \leq 2C_{46} + \pi^{-1} C_{51}$, follows directly from (45), (47), (56) and (29). ∎

4. Commentaries and concluding remarks

Let us compare Theorems 2 and 3 with the relevant results for $d \neq 2$. It is well known that the lowest possible order of growth for $N(-\gamma; A_{\alpha V})$, $\gamma \geq 0$, in the d-dimensional case is

$$N(-\gamma; A_{\alpha V}) = O(\alpha^{d/2}) \ , \qquad \alpha \longrightarrow \infty. \tag{57}$$

We call *regular* any estimate of $N(-\gamma; A_{\alpha V})$ giving just this order of growth. The regular estimate for $d \geq 3$, $\gamma = 0$, was first obtained by Rozenblum [R1]:

$$N(0; A_{\alpha V}) \leq C(d) \alpha^{d/2} \int_{\mathbb{R}^d} V^{d/2} dx \ , \quad \forall \alpha > 0 \ , \quad d \geq 3. \tag{58}$$

Unaware of Rozenblum's paper, Simon [S1] obtained, also for $d \geq 3$, some regular estimate for a more restricted class of potentials V. The inequality (58) was stated in [S1] as a conjecture. Then it was justified in [C], [L1], by methods essentially different from each other and also from the original method of [R1].

It turns out that the estimate (58) is sharp both in α and in function classes for V: it was proved in [R1] that if $V \in L_{1,loc}(\mathbb{R}^d)$, $d \geq 3$, and (57) takes place for $\gamma = 0$, then $V \in L_{d/2}$. The estimate for $\gamma > 0$ following from (58) and from the obvious inequality $N(-\gamma; A_{\alpha V}) \leq N(0; A_{\alpha V})$ turns out also to be sharp both in α and V.

In contrast to the case $d \geq 3$, for $d = 1$ the regular estimates for $\gamma > 0$ and $\gamma = 0$ substantially differ from each other. Denote $I_i(h) = [(i-1)h, ih]$, $i \in \mathbb{Z}$, $h > 0$, and $\mathcal{J}_i = \{x \in \mathbb{R}: 2^{i-1} \leq |x| \leq 2^i\}$, $i > 0$. Then for $d = 1$

$$N(-\gamma; A_{\alpha V}) \leq C\alpha^{\frac{1}{2}} \sum_{i\in\mathbb{Z}} \Big(\int_{I_i(\gamma^{-1})} V dx\Big)^{\frac{1}{2}}, \quad \alpha, \gamma > 0; \tag{59}$$

$$N(0; A_{\alpha V}) \leq 1 + C\alpha^{\frac{1}{2}}\Big(\Big(\int_{-1}^{1} V dx\Big)^{\frac{1}{2}} + \sum_{i\in\mathbb{N}}\Big(\int_{\mathcal{J}_i} |x|V(x)dx\Big)^{\frac{1}{2}}\Big), \quad \alpha > 0. \tag{60}$$

So, some resemblance exists between the estimates (25) and (59) concerning the case $\gamma > 0$. For $\gamma = 0$, a significant difference between the estimates (31) and (60) arises, due to the presence of the term $\|\nu(V)\|_{1,w}$ in (31). Note that the appearance of an additive constant in (31) and (60) is unavoidable because $\lambda = 0$ is the resonance point for the one- and two-dimensional Laplacian. The estimate (59) was obtained in [BB] and (60) in [BS3]. Both estimates are sharp in α and "almost sharp" in function classes for V; namely, suppose that for a fixed $\gamma \geq 0$, $N(-\gamma, A_{\alpha V}) = O(\alpha^{\frac{1}{2}})$. Then it is easy to show that the sequence of integrals appearing in (59) (for $\gamma > 0$) or in (60) (for $\gamma = 0$) has to belong to the "weak $\ell_{\frac{1}{2}}$-space" $\ell_{\frac{1}{2},w}$. To check it, one can use the same arguments as in the proofs of Theorems 5.2 and 6.2 of [BS3]. So, the difference between the function classes involved in (59) and (60) and the sharp ones guaranteeing the regular estimate (57) for $d = 1$, is no greater than the difference between $\ell_{\frac{1}{2}}$ and $\ell_{\frac{1}{2},w}$.

Our estimates (25) and (31) for $d = 2$ are also sharp in α but it is much more difficult to find out whether they are sharp in function classes as well. A complete description of the class M_d of weights guaranteeing the boundedness in $H^1(Q)$ of the quadratic functional $\int_Q V|u|^2dx$, $Q = Q^d$, was given by Maz'ya (see [M], No. 2.3.3). The sharpness of our starting spectral estimate – Theorem 4 – would mean that if $V \in \mathrm{M}_2$, the operator $T_{Q^2,V}$ is compact and $n(\lambda; T_{Q^2,V}) = O(\lambda^{-1})$, then $V \in L_{\mathcal{B}}(Q^2)$. It is not quite clear at the moment whether Theorem 4 is sharp in such a "strong" form. However, it follows from the result of [HMT] that

Theorem 4 can not be refined in terms of Orlicz spaces. Note that if we suppose that Theorem 4 is sharp, then Theorem 2 is "almost sharp" in the same sense as the estimate (59).

The problem of the sharpness of the estimate (31) is complicated by the presence of two terms $\|\mu(V)\|_1$, $\|\nu(V)\|_{1,w}$ of a quite different nature on its right hand side. The last term must be present in any sharp estimate (possibly in some implicit form) because it corresponds to the spectrum of an operator generated by the quadratic form (24) restricted to a subspace. It was shown in [BS3], Theorem 6.1, that the condition of Proposition 4 is not only sufficient, but also necessary to guarantee the order $n(\lambda; M_W) = O(\lambda^{-1})$, $\lambda \to +0$. But it may turn out that the estimate (46) is not sharp. Indeed, the passage from $\mathcal{G}_1$ to $\sum_i \oplus \hat{H}^1(\Omega_i)$, used in the proof of Corollary 1′, seems to be crude.

The passage from the estimate (31) to (32) (after the formulation of Theorem 3) was based on the inequality (30). A more direct way to obtain (32) could be used instead: namely, the estimate

$$N(0; A_{\alpha V}^{(0)}) \leq 1 + \alpha \int |\ell n|x||V(x)dx$$

is a straightforward consequence of the well known Bargmann estimate (see e.g. [RS]) for the 1-dimensional Schrödinger operators.

We followed the approach proposed in [R1], [R2] (or, to be more precise, its simplified version given in [BS2]) when proving Theorem 1. The use of Fourier series with respect to φ in the preparatory part of the proof of Theorem 3 was borrowed from the paper by Egorov and Kondrat′ev [EK]. They used Fourier series in a somewhat different way and did not get any regular estimates for $N(0; A_{\alpha V})$.

It may happen that the integral in (58) (for $d \geq 3$) diverges, or the quantities appearing in (25), (31) (for $d = 2$) and (59), (60) (for $d = 1$) are equal to ∞, but nevertheless the negative spectrum of $A_{\alpha V}$ is discrete. Then by necessity $N(-\gamma; A_{\alpha V})$ has a non-regular order of growth. These cases are also of some interest for applications. In particular, Proposition 4 concerns just one of such cases. The paper [BS3] was devoted to the systematic investigation of "intermediate" estimates of this type. Similar results can also be obtained for $d = 2$ using Theorems 2 and 3.

Among other possible extensions of the results presented, the applications to the spectral estimates for the higher order PDE must be mentioned first. A statement similar to Theorem 2 for the operator $(-\Delta)^\ell - \alpha V$ on $\mathbb{R}^d$, $2\ell = d$, can be easily obtained by the same approach. However one meets with some additional difficulties when trying to extend the result of Theorem 3. We are going to consider these and some other related problems in a separate paper.

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